

On the Cuts of Scattering Amplitudes

Pierpaolo Mastrolia^{a*}

^aCERN PH-TH, CH-1211 Geneva 23, Switzerland

The use of complex analysis for computing one-loop scattering amplitudes is naturally induced by generalised unitarity-cut conditions, fulfilled by complex values of the loop variable. We report on two techniques: the cut-integration with spinor-variables as contour integrals of rational functions; and the use of the Discrete Fourier Transform to optimize the reduction of tensor-integrals to master scalar integrals.

1. GENERALISED UNITARITY

The application of unitarity as an on-shell method of calculation [1,2] is based on the principles that products of on-shell tree-level amplitudes produce functions with the correct branch cuts in all channels; and that any one-loop amplitude is expected to be expressed, by Passarino-Veltman reduction, as a linear combination of scalar master integrals (MI's), that are characterised by their own, leading and subleading, singularities [3].

Dimensionally-regulated amplitudes can be decomposed in terms of MI's in shifted-dimensions as,

$$\mathcal{A}_N^{(4-2\epsilon)} = \sum_{n, 0 \leq j \leq j_{\max}} c_{nj} \times \mathcal{M}_n^{(4-2\epsilon+2j)}, \quad (1)$$

where $\mathcal{A}_N^{(4-2\epsilon)}$ is any N -point amplitude in D -dimensions (being $D = 4 - 2\epsilon$), and $\mathcal{M}_n^{(4-2\epsilon+2j)}$ is a n -point MI's in $(4 - 2\epsilon + 2j)$ -dimensions, with $n \in \{0, 2, 3, 4, \dots, N\}$, and j_{\max} depending on the process. In Eq.(1), the coefficients c_{nj} do not depend on D , and the whole D -dependence is embedded in the definition of \mathcal{M} 's [4,2],

$$\mathcal{M}_n^{(4-2\epsilon+2j)} = \frac{\Omega_{-1-2\epsilon}}{\Omega_{-1-2\epsilon+2j}} (4\pi)^j I_n^{(4-2\epsilon+2j)}, \quad (2)$$

$$\Omega_k = 2\pi^{\frac{k+1}{2}} \Gamma^{-1}((k+1)/2), \quad (3)$$

where Ω_k is the generalised solid-angle, and $I_n^{(4-2\epsilon+2j)}$ is the scalar n -point function in shifted

dimensions. The decomposition (1) could be further simplified with the help of recurrence relations linking higher-point integrals to lower-point ones [5]. From Eq.(1), it is clear that the computation of the amplitude requires the knowledge of two types of ingredients: the MI's, and their coefficients. In the following, we focus on the determination of the latter ones.

The principle of a unitarity-based method is the extraction of the *rational* coefficients, c_{nj} , by matching the multiparticle cuts of the amplitude onto the corresponding cuts of the MI's. By considering [6,2,7] the splitting of L_D , the loop momentum in $(4 - 2\epsilon)$ -dimension, into its four-dimensional component, L , and its orthogonal complement $L_{(-2\epsilon)}$, as $L_D \equiv L + L_{(-2\epsilon)}$ (with $L_D^2 \equiv L^2 + L_{(-2\epsilon)}^2$ and $L_{(-2\epsilon)}^2 \equiv -\mu^2$), the D -dimensional integration measure can be written as a convolution of a four-dimensional integration and an integration on μ^2 ,

$$\int d^D L_D = \int d^{-2\epsilon} \mu \int d^4 L = \quad (4)$$

$$= \Omega_{-2\epsilon} \int d\mu^2 (\mu^2)^{-1-\epsilon} \int d^4 L. \quad (5)$$

By applying the above splitting to both sides of Eq.(1), the four-dimensional kernel of the amplitude, $\mathcal{A}_N^{(4)}$, can be read as expressed in terms of four-dimensional n -point MI's, $I_n^{(4)}$,

$$\mathcal{A}_N^{(4)}(\mu^2) = \sum_{n,j} c_{nj} \times (\mu^2)^j \times I_n^{(4)}(\mu^2). \quad (6)$$

We notice that the μ^2 -dependence of $\mathcal{A}_N^{(4)}$ is due to the presence of μ^2 in all the denominators of

*email: Pierpaolo.Mastrolia@cern.ch

$I_n^{(4)}$, as additional mass-term, and to the polynomial coefficients, $c_{nj} \times (\mu^2)^j$. It is therefore evident that to find out c_{nj} , it is sufficient to compute the μ^2 -polynomials [14] that are the coefficients of the four-dimensional MI's, $I_n^{(4)}$, in the decomposition (6).

$I_n^{(4)}$ are functions determined by their own branch-cuts. Generalised unitarity is a very effective tool to extract the rational coefficients of functions by exploiting their singularity structure, which is accessed by imposing (on-shellness) cut-condition to propagating particles,

$$(q^2 - m^2 + i0)^{-1} \rightarrow (2\pi i) \delta^{(+)}(q^2 - m^2). \quad (7)$$

In general, the fulfillment of multiple-cut conditions requires loop momenta with complex components. Since the loop momentum, L , in Eq.(6) has four components, the effect of the cut-conditions is to freeze some of its components, when not all, according to the number of the cuts. With the *quadruple*-cut [8] the loop momentum is completely frozen, yielding the algebraic determination of the coefficients of $I_n^{(4)}$, ($n \geq 4$); the spinorial integration of the *double*-cut [9,10,11,15] and *triple*-cut [15,16,17] lead to the reconstruction of $I_2^{(4)}$ - and $I_3^{(4)}$ -coefficients; while the coefficients of $I_0^{(4)}$ are detected by *single*-cut. In cases where fewer than four denominators are cut, the loop momentum is not frozen: the free-components are left over as integration variables of the phase-space. We will discuss two strategies for dealing with the degrees of freedom represented by those variables: *i*) analytic integration of the phase-space with spinor variables [9,10,11,12,13,14]; *ii*) algebraic decomposition of the integrand by means of the Discrete Fourier Transform (DFT) [23].

2. DOUBLE-CUT AND SPINORS

By using the splitting of the loop variables as above, the D -dimensional double-cut in the P -channel of any amplitude, *i.e.* the double-cut of the *l.h.s* of Eq.(1), can be written as a convolution of a four-dimensional double-cut and an integra-

tion on μ^2 ,

$$\Delta(\mathcal{A}_N^{(D)}) = \int d\mu^{-2\epsilon} \Delta(\mathcal{A}_N^{(4)}), \quad (8)$$

where $\Delta(\mathcal{A}_N^{(4)})$ is the double-cut of the *l.h.s* of Eq.(6),

$$\Delta(\mathcal{A}_N^{(4)}) = \int d^4 L \delta^{(+)}(L^2 - M_1^2 - \mu^2) \times \delta^{(+)}((L-P)^2 - M_2^2 - \mu^2) A_1^{\text{tree}} A_2^{\text{tree}}, \quad (9)$$

with A_i^{tree} being the tree-amplitudes sewn in the cut. We found it convenient to decompose [11] the four-dimensional loop variable, L , in terms of a massless momentum ℓ , and the momentum across the cut, P ,

$$L_\nu = t \ell_\nu + z_0 P_\nu, \quad (10)$$

$$t = (1 - 2z_0)P^2 / \langle \ell | P | \ell \rangle, \quad (11)$$

$$z_0 = (P^2 + M_1^2 - M_2^2 - \sqrt{\lambda - 4\mu^2}) / 2P^2, \quad (12)$$

$$\lambda = (P^2)^2 + (M_1^2)^2 + (M_2^2)^2 - 2P^2 M_1^2 - 2P^2 M_2^2 - 2M_1^2 M_2^2, \quad (13)$$

with z_0 being the anomalous threshold, and λ , the Källén function. With the above transformation, $\Delta(\mathcal{A}_N^{(4)})$ can be written in terms of spinor-variables [9,10], $|\ell\rangle$ and $|\ell]$ (associated to the massless momenta, ℓ , through $\ell = |\ell\rangle[\ell|$), and can be cast as a sum of terms whose general structure reads,

$$\Delta(\mathcal{A}_N^{(4)}) = \sum_i \Delta_i, \quad \Delta_i = \int \langle \ell d\ell | [\ell d\ell] \mathcal{I}_i, \quad (14)$$

$$\mathcal{I}_i = \rho_i(|\ell\rangle) \frac{[\eta \ell]^n}{\langle \ell | P_1 | \ell \rangle^{n+1} \langle \ell | P_2 | \ell \rangle}, \quad (15)$$

where P_1 and P_2 can either be equal to the cut-momentum P , or be a linear combination of external vectors; and where the ρ_i 's depend solely on one spinor flavour, say $|\ell\rangle$ (and not on $|\ell|$), and may contain poles in $|\ell\rangle$. We give as understood the dependence of \mathcal{I}_i on μ^2 , through the variable z_0 . The explicit form of the vectors P_1 and P_2 in (15) is determining the nature of the double-cut, logarithmic or not, and correspondingly the topology of the diagram which is associated to: if $P_1 = P_2 = P$,

$$\mathcal{I}_i = \rho_i(|\ell\rangle) [\eta \ell]^n / \langle \ell | P | \ell \rangle^{n+2}, \quad (16)$$

and the result will be non-logarithmic, hence corresponding to the cut of a 2-point function with external momentum P ; if $P_1 = P$, $P_2 \neq P$ or $P_1 \neq P_2 \neq P$, one proceeds by introducing a Feynman parameter, to write \mathcal{I}_i as,

$$\mathcal{I}_i = (n+1) \int_0^1 dx (1-x)^n \frac{\rho_i(|\ell\rangle) [\eta \ell]^n}{\langle \ell | R | \ell \rangle^{n+2}}, \quad (17)$$

$$\not{R} = x \not{P}_1 + (1-x) \not{P}_2, \quad (18)$$

and (because of the parametric integral) the result is logarithmic, hence containing the cut of a linear combination of n -point functions with $n \geq 3$. The spinorial structure of Eq. (16) and Eq. (17) is the same. Therefore, we discuss the spinor integration of the latter, because it is more general.

2.1. Contour Integrals

One can proceed with a change of variables [18], decomposing $|\ell\rangle$ and $|\ell\rangle$ into two arbitrary massless momenta, p and q (light-cone decomposition),

$$\forall p, q : q^2 = p^2 = 0, \quad (19)$$

$$|\ell\rangle \equiv |p\rangle + z|q\rangle \quad |\bar{\ell}\rangle \equiv |p\rangle + \bar{z}|q\rangle \quad (20)$$

$$\langle \ell | d\ell \rangle [\bar{\ell} | d\bar{\ell}] = -\langle q | p | q \rangle dz d\bar{z}. \quad (21)$$

Its effect on Δ_i reads,

$$\Delta_i = (n+1) \int_0^1 dx (1-x)^n \langle q | p | q \rangle \times \oint dz d\bar{z} \rho_i(z) \frac{([\eta p] + \bar{z}[\eta q])^n}{\chi^{n+2}(z, \bar{z})}, \quad (22)$$

$$\chi(z, \bar{z}) = \langle p | R | p \rangle + z \langle q | R | p \rangle + \bar{z} \langle p | R | q \rangle + z \bar{z} \langle q | R | q \rangle. \quad (23)$$

One may observe that the z - \bar{z} -integrand can be written as a total derivative with respect to \bar{z}

$$\Delta_i = \int_0^1 dx (1-x)^n \langle q | p | q \rangle \times \oint dz d\bar{z} \frac{d}{d\bar{z}} \left\{ \rho_i(z) \frac{([\eta p] + \bar{z}[\eta q])^{n+1}}{\xi(z) \chi^{n+1}(z, \bar{z})} \right\} \quad (24)$$

with $\xi(z) = (\langle p | P | p \rangle + z \langle q | P | p \rangle)$, so that the spinor integration has been turned into a contour integral of a rational function in z ,

$$\Delta_i = \int_0^1 dx (1-x)^n \langle q | p | q \rangle \times \oint dz \left\{ \rho_i(z) \frac{([\eta p] + \bar{z}[\eta q])^{n+1}}{\xi(z) \chi^{n+1}(z, \bar{z})} \right\}. \quad (25)$$

The z -integral can be performed by Cauchy's residue theorem, summing the residues at the poles in z (substituting as well $\bar{z} = z^*$). There are two sources of poles to account for: *i*) the poles contained in $\rho_i(z)$; *ii*) the pole due to $\xi(z)$, whose value is

$$z_\xi = -\langle p | P | q \rangle / \langle q | P | q \rangle. \quad (26)$$

To complete the integration of Δ_i in (25), one has to perform the parametric integration which is finally responsible for the appearance of logarithmic terms in the double-cut. On the contrary, the spinorial integration in (16) would generate a contribution without branch-cuts. We remark that the role of $|\ell\rangle$ and $|\bar{\ell}\rangle$ in the integration could be interchanged.

At the end of the phase-space integration, by adding up all the Δ_i 's, one finally gets a result whose structure is

$$\Delta(\mathcal{A}_N^{(4)}) = \sum_{2 \leq n, j} c_{nj} \times (\mu^2)^j \times \Delta(I_n^{(4)}), \quad (27)$$

corresponding to the double-cut of Eq.(6). Out of (27), it is possible to extract the polynomial coefficients, $c_{nj} \times (\mu^2)^j$ ($n \geq 2$): the coefficient of $I_0^{(4)}$, the tadpoles, cannot be detected within the double-cut, and their determination should be provided by independent informations on the amplitude. We recall that the μ^2 -dependence of the coefficients originates from the understood presence of z_0 , given in Eq.(12).

We observe that a proper choice of the momenta p and q , entering the change of variables (20), can simplify dramatically the calculation. For instance, they determine the value of the z_ξ -pole, given in Eq.(26): given q_μ , and the cut-momentum P_μ , the choice $p_\mu \equiv P_\mu - q_\mu \times P^2 / \langle q | P | q \rangle$, would yield $z_\xi = 0$.

The phase-space integration just discussed was used successfully for an analytic computation of non-trivial one-loop corrections. In particular, its four-dimensional (massless) version [9,10,16] has been applied to complete the non-supersymmetric cut-constructible term of the six-gluon amplitude in QCD [10], to compute the six-photon amplitude in QED [20,19], and the cut-constructible term of a general MHV amplitudes involving a

Higgs plus n -gluons in QCD (in the heavy-top limit) [21].

Recently, the efficiency of spinor integration has been pushed to achieve closed analytic forms for the generating formulas of the coefficients of $I_n^{(4)}$ ($2 \leq n \leq 4$), for an arbitrary massive process [13,14], which together with $I_0^{(4)}$ constitute a basis of functions in four-dimensions, hence, due to the relation among Eq.(6) and Eq.(1), in D -dimensions. The formulas presented in [13,14] - too long to be shown here - can be evaluated, without performing any integration, by specializing the value of input variables that are specific to the initial cut-integrand as assembled from tree-level amplitudes.

We have as well recently released the package **SOM** (Spinors @ Mathematica)[22], that implements the spinor-helicity formalism in Mathematica. The package allows the use of complex-spinor algebra along with the multi-purpose features of Mathematica, and it is suitable for the algebraic manipulation and integration of products of tree amplitudes with complex spinors sewn in generalised unitarity-cuts.

3. OPTIMIZED REDUCTION

As an alternative to any phase-space integration, in [24,25] there was proposed a very efficient method for the reconstruction of the coefficients in the decomposition (6). In what follows, I limit the discussion to the so called cut-constructible term of a scattering amplitude, that corresponds to the poly-logarithmic structure arising when Eq.(6) is evaluated at $\mu^2 = 0$. I will sketch the reconstruction of the complete μ^2 -dependence [26,27,28,29] at the end of the section. The by-now known as OPP-reduction allows the numerical reconstruction of c_{n0} , by solving a system of algebraic equations that are obtained by: *i*) the numerical evaluation of the *integrand* at explicit values of the loop-variable, on the one side; *ii*) and the knowledge of the most general *polynomial* structure of the *integrand* itself [30], on the other one. The values of the loop momentum used for the numerical evaluation of the integrand are chosen among the set of solutions of the multiple-cut conditions, *i.e.* the solutions of the system of

equations obtained by imposing the vanishing of the cut-denominators.

3.1. OPP-Reduction

The starting point of the OPP reduction method [24,25] is the general expression for the *integrand* of a generic m -point one-loop amplitude that can be written as

$$A_m(q) = \frac{N(q)}{D_0 D_1 \cdots D_{m-1}}, \quad (28)$$

$$D_i = (q + p_i)^2 - m_i^2, \quad p_0 \neq 0, \quad (29)$$

where $N(q)$ is the four-dimensional numerator of the amplitude.² The main formula of the OPP-algorithm is the expression of $N(q)$ in terms of the denominators D_i ,

$$N(q) = \sum_{\alpha=1}^4 \Delta_{\alpha}(q) \quad (30)$$

with

$$\Delta_4(q) = \sum_{i < j < k < \ell}^{m-1} \left[d_{ijkl} + \tilde{d}_{ijkl}(q) \right] \prod_{\beta \neq ijk\ell}^{m-1} D_{\beta}, \quad (31)$$

$$\Delta_3(q) = \sum_{i < j < k}^{m-1} [c_{ijk} + \tilde{c}_{ijk}(q)] \prod_{\beta \neq ijk}^{m-1} D_{\beta}, \quad (32)$$

$$\Delta_2(q) = \sum_{i < j}^{m-1} [b_{ij} + \tilde{b}_{ij}(q)] \prod_{\beta \neq ij}^{m-1} D_{\beta}, \quad (33)$$

$$\Delta_1(q) = \sum_i^{m-1} [a_i + \tilde{a}_i(q)] \prod_{\beta \neq i}^{m-1} D_{\beta}. \quad (34)$$

By inserting (30) back in (28), one exposes the multi-pole nature of A_m . The coefficients of the multi-pole expansion can be further split in two pieces: a piece that still depends on q , parametrized by $\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}$, that vanishes upon integration, and a piece that does not depend on q , parametrized as d, c, b, a . Such a separation is always possible, as shown in [24], and, with this choice, the latter set of coefficients corresponds to the ensemble of the coefficients of $I_n^{(4)}(\mu^2 = 0)$, ($n \in \{0, 2, 3, 4\}$): a, b, c, d in (30) correspond respectively to $c_{00}, c_{20}, c_{30}, c_{40}$ in (6).

² $A_m(q)$ is the integrand of $\mathcal{A}_m^{(4)}(\mu^2 = 0)$, defined in Eq.(6)

3.2. Top-Down System

The goal of the algorithm is reduced to the algebraical problem of fitting the coefficients d, c, b, a by evaluating the function $N(q)$ a sufficient number of times, at different values of q , and then inverting the system. Accordingly, let us define the following functions,

$$R_{ijkl}(q) \equiv N(q) \left(\prod_{\beta \neq ijk\ell}^{m-1} D_\beta \right)^{-1}, \quad (35)$$

$$R'_{ijk}(q) \equiv (N(q) - \Delta_4(q)) \left(\prod_{\beta \neq ijk}^{m-1} D_\beta \right)^{-1}, \quad (36)$$

$$R''_{ij}(q) \equiv (N(q) - \sum_{\alpha=3}^4 \Delta_\alpha(q)) \left(\prod_{\beta \neq ij}^{m-1} D_\beta \right)^{-1}, \quad (37)$$

$$R'''_i(q) \equiv (N(q) - \sum_{\alpha=2}^4 \Delta_\alpha(q)) \left(\prod_{\beta \neq i}^{m-1} D_\beta \right)^{-1}. \quad (38)$$

We as well define as $\{q\}_{ijkl}$ the set of the solutions of $D_i = D_j = D_k = D_\ell = 0$. Having defined our setup, from Eq.(30) we can derive the following sets of equations:

$$\left[R_{ijkl}(q) = d_{ijkl} + \tilde{d}_{ijkl}(q) \right]_{q \in \{q\}_{ijkl}}, \quad (39)$$

$$\left[R'_{ijk}(q) = c_{ijk} + \tilde{c}_{ijk}(q) \right]_{q \in \{q\}_{ijk}}, \quad (40)$$

$$\left[R''_{ij}(q) = b_{ij} + \tilde{b}_{ij}(q) \right]_{q \in \{q\}_{ij}}, \quad (41)$$

$$\left[R'''_i(q) = a_i + \tilde{a}_i(q) \right]_{q \in \{q\}_i}, \quad (42)$$

which *must* be solved necessarily in cascade, top-down: in Eq.(39), $N(q)$ is a known quantity, namely an input of the algorithm; but the *l.h.s* of each other equation becomes a known quantity (numerically evaluable), only after solving the equation which precedes it.

3.3. Polynomial Structures and DFT

An important observation is due. The *r.h.s* of each of the equations (39)-(42) is a *polynomial* function. Without presenting their explicit expressions (see [23] for the detailed presentation), the general structure is the following: the variables are the components of q not-frozen by the cut-conditions; the degree is known; while the

coefficients are the *unknowns* to be determined. The problem to be tackled is thus a well known mathematical subject: *polynomial interpolation*. In order to find out the coefficients of a polynomial, one can avoid the numerical inversion of a system, which is a very delicate operation, due to the possibility of a vanishing determinant in critical kinematic regions.

The Discrete Fourier Transform (DFT) is a very efficient tool to extract the coefficients of a polynomial, by evaluating it at special values of the variables [13,23,31]. Let us show how it works in the case of a polynomial of degree n in one variable, x , defined as,

$$P_n(x) = \sum_{\ell=0}^n c_\ell x^\ell. \quad (43)$$

At the first step, one generates the set of discrete values $P_{n,k}$ ($k = 0, \dots, n$),

$$P_{n,k} \equiv P_n(x_k) = \sum_{\ell=0}^n c_\ell \rho^\ell e^{-2\pi i \frac{k}{(n+1)} \ell}, \quad (44)$$

by sampling $P_n(x)$ at $(n+1)$ equidistant points on the ρ -circle,

$$x_k = \rho e^{-2\pi i \frac{k}{(n+1)}}. \quad (45)$$

At the second step, using the orthogonality

$$\sum_{j=0}^{n-1} e^{2\pi i \frac{k}{n} j} e^{-2\pi i \frac{k'}{n} j} = n \delta_{kk'}, \quad (46)$$

one can obtain the coefficient c_ℓ simply by *projection*,

$$c_\ell = \frac{\rho^{-\ell}}{n+1} \sum_{k=0}^n P_{n,k} e^{2\pi i \frac{k}{(n+1)} \ell}. \quad (47)$$

In fact, the *r.h.s* of Eq.(42) is a degree-1 polynomial in a single variable, whose coefficients are easily determined by the semi-sum and the semi-difference of two numerical values of R . But the *r.h.s* of Eqs.(40)-(42) are multivariate polynomials of higher degree. To find out their coefficients we used a modified DFT, that is a Fast Fourier Transform-like algorithm, suitable to minimize the number of the numerical calls respectively of R', R'' , and R''' , being exactly the same as the

number of the unknowns, and to avoid the kinematical singularities emerging at the vanishing of the circle-radius ρ . In so doing, one can determine all the unknown coefficients, among which the O^{th} -order ones, respectively d_{ijkl} , c_{ijk} , b_{ij} , a_i , correspond to the coefficients of the MI's in four-dimension.

For the reconstruction of the complete μ^2 -dependence of the coefficients in Eq.(6), the decomposition (30), must be slightly extended to account for the presence of μ^2 [28,29]. The starting point, in this case, is $A_m^{(4)}(\mu^2)$, which contains a numerator $N(q, \mu^2)$ and denominators $\bar{D}_i = D_i - \mu^2$. The reduction proceeds exactly as above, with the difference Eqs.(39)-(42) containing an extra dependence on μ^2 . Since the μ^2 -dependence is still polynomial, one can use the DFT also in this case, having to deal with R, R', R'' , and R''' with μ^2 as additional variable [13]. The flexibility of the projection-procedure hereby presented extends its range of applicability to tackle the determination of the coefficients of polynomial structures wherever should this issue occur. We finally remark that the parametrization of the free (integration) variables as complex unitary phases yields as well a very effective performance of Cauchy's residue theorem within the contexts of factorization- and unitarity-based methods, where the on-shellness properties are naturally captured by polar structures of complex phases.

REFERENCES

1. Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217.
2. Z. Bern and A. G. Morgan, Nucl. Phys. B **467** (1996) 479.
3. Z. Bern, L. J. Dixon and D. A. Kosower, Annals Phys. **322**, 1587 (2007).
4. Z. Bern and G. Chalmers, Nucl. Phys. B **447**, 465 (1995).
5. Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299 [Erratum-ibid. B **318** (1993) 649].
6. G. Mahlon, [arXiv:hep-ph/9311213].
7. A. Brandhuber, S. McNamara, B. J. Spence and G. Travaglini, JHEP **0510**, 011 (2005).
8. R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **725**, 275 (2005).
9. R. Britto, E. Buchbinder, F. Cachazo and B. Feng, Phys. Rev. D **72**, 065012 (2005).
10. R. Britto, B. Feng and P. Mastrolia, Phys. Rev. D **73**, 105004 (2006).
11. C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, Phys. Lett. B **645**, 213 (2007); JHEP **0703**, 111 (2007); R. Britto and B. Feng, Phys. Rev. D **75**, 105006 (2007).
12. R. Britto and B. Feng, JHEP **0802**, 095 (2008).
13. R. Britto, B. Feng and P. Mastrolia, arXiv:0803.1989 [hep-ph].
14. R. Britto, B. Feng and G. Yang, arXiv:0803.3147 [hep-ph].
15. D. Forde, Phys. Rev. D **75**, 125019 (2007).
16. P. Mastrolia, Phys. Lett. B **644**, 272 (2007).
17. N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, JHEP **0804**, 038 (2008).
18. R. Britto, *talk*, Workshop HP2, Zürich, Switzerland, September 6-9, 2006.
19. T. Binoth, G. Heinrich, T. Gehrmann and P. Mastrolia, Phys. Lett. B **649**, 422 (2007).
20. T. Binoth, J. P. Guillet and G. Heinrich, JHEP **0702**, 013 (2007).
21. E. W. N. Glover, P. Mastrolia and C. Williams, arXiv:0804.4149 [hep-ph].
22. D. Maitre and P. Mastrolia, arXiv:0710.5559 [hep-ph].
23. P. Mastrolia, G. Ossola, C. G. Papadopoulos and R. Pittau, arXiv:0803.3964 [hep-ph].
24. G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007).
25. G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **0707**, 085 (2007).
26. R. Pittau, Comput. Phys. Commun. **104**, 23 (1997); **111** (1998) 48.
27. R. K. Ellis, W. T. Giele and Z. Kunszt, JHEP **0803** (2008) 003.
28. W. T. Giele, Z. Kunszt and K. Melnikov, JHEP **0804** (2008) 049.
29. G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **0805** (2008) 004.
30. F. del Aguila and R. Pittau, JHEP **0407** (2004) 017; R. Pittau, arXiv:hep-ph/0406105.
31. C. F. Berger *et al.*, arXiv:0803.4180 [hep-ph].